

# WEAK\*-CLOSEDNESS OF SUBSPACES OF FOURIER-STIELTJES ALGEBRAS AND WEAK\*-CONTINUITY OF THE RESTRICTION MAP

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*Dedicated to Professor Elmar Thoma on the occasion of his seventieth birthday*

ABSTRACT. Let  $G$  be a locally compact group and  $B(G)$  the Fourier-Stieltjes algebra of  $G$ . We study the problem of how weak\*-closedness of some translation invariant subspaces of  $B(G)$  is related to the structure of  $G$ . Moreover, we prove that for a closed subgroup  $H$  of  $G$ , the restriction map from  $B(G)$  to  $B(H)$  is weak\*-continuous only when  $H$  is open in  $G$ .

## INTRODUCTION

Let  $G$  be a locally compact group, and let  $B(G)$  be the Fourier-Stieltjes algebra of  $G$  as defined by Eymard [8]. Recall that  $B(G)$  is the linear span of all continuous positive definite functions on  $G$  and can be identified with the Banach space dual of  $C^*(G)$ , the group  $C^*$ -algebra of  $G$ . The space  $B(G)$ , with the norm as dual of  $C^*(G)$ , is a commutative Banach  $*$ -algebra with pointwise multiplication and complex conjugation. The Fourier algebra  $A(G)$  of  $G$  is the closed  $*$ -subalgebra of  $B(G)$  generated by the functions in  $B(G)$  with compact support. In particular,  $A(G)$  is contained in  $C_0(G)$ , the algebra of complex valued continuous functions on  $G$  vanishing at infinity. As is well known  $A(G)$  is weak\*-dense in  $B(G)$  if and only if  $G$  is amenable. In [3] translation invariant  $*$ -subalgebras  $A$  of  $B(G)$  were studied, and it was shown that if such  $A$  is weak\*-closed and point separating, then it must contain  $A(G)$ . However, apart from this, very little seems to be known about weak\*-closed subspaces of  $B(G)$ .

The first purpose of this paper is to investigate the relation between weak\*-closedness of certain interesting norm-closed translation invariant subspaces of  $B(G)$  and the structure of  $G$ . Secondly, we solve the problem of when, for a closed subgroup  $H$  of  $G$ , the restriction map from  $B(G)$  to  $B(H)$  is weak\*-continuous.

A brief outline of the paper is as follows. In Section 2 we establish for almost connected locally compact groups  $G$  the relation between weak\*-closedness of  $B_0(G) = B(G) \cap C_0(G)$  in  $B(G)$  and the structure of  $G$  (Theorem 2.10). The key result is that for a connected Lie group  $G$ ,  $B_0(G)$  is weak\*-closed in  $B(G)$  if and only if  $G$  is a reductive Lie group with compact centre and Kazhdan's property (T).

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If  $G$  is a compact group, then for any unitary representation  $\pi$  of  $G$ , the Fourier space  $A_\pi(G)$  associated to  $\pi$  (see [1] for the definition and properties of  $A_\pi(G)$ ) is weak\*-closed in  $B(G)$ . Note that  $A(G) = A_{\lambda_G}(G)$ , where  $\lambda_G$  denotes the left regular representation of  $G$ . In Theorem 3.6 we shall show that, conversely, if  $G$  contains an almost connected open normal subgroup and  $A(G)$  is weak\*-closed in  $B(G)$ , then  $G$  is compact. We also give a characterization of compactness of  $G$  in terms of the weak\* and the norm topologies on the unit sphere of  $B(G)$  (Theorem 3.9).

Besides the left regular representation, one of the most interesting representations of a locally compact group  $G$  is the conjugation representation  $\gamma_G$  of  $G$  on  $L^2(G)$ . In contrast to  $A(G)$ ,  $A_{\gamma_G}(G)$  need not be a subalgebra and it can at best determine the structure of  $G/Z(G)$ , where  $Z(G)$  denotes the centre of  $G$ . We prove that if  $G$  is a Lie group with countably many connected components and  $A_{\gamma_G}(G)$  is weak\*-closed in  $B(G)$ , then  $G/Z(G)$  is compact (Theorem 4.8).

Let  $H$  be a closed subgroup of an arbitrary locally compact group  $G$ . Clearly, the restriction map  $B(G) \rightarrow B(H)$  is continuous for the weak\*-topologies whenever  $H$  is open. In the final section 5 we succeed in showing that conversely weak\*-continuity of the restriction map forces  $H$  to be open in  $G$ .

## 1. PRELIMINARIES

Throughout this paper,  $G$  denotes a locally compact group with a fixed left Haar measure  $dx$  and modular function  $\Delta$ ,  $L^1(G)$  the convolution algebra of integrable functions on  $G$  and  $C^*(G)$  the group  $C^*$ -algebra of  $G$ . The Fourier-Stieltjes algebra  $B(G)$  is the Banach space dual of  $C^*(G)$  and as such carries the weak\*-topology ( $w^*$ -topology, for short)  $\sigma(B(G), C^*(G))$ . The basic reference on Fourier and Fourier-Stieltjes algebras is [8].

Next, we have to introduce some notation from representation theory. We use the same letter, for example  $\pi$ , for a unitary representation of  $G$  and for the corresponding \*-representation of  $C^*(G)$ .  $\mathcal{H}_\pi$  will always denote the Hilbert space of  $\pi$  and  $\ker \pi$  the  $C^*$ -kernel of  $\pi$ . If  $S$  and  $T$  are sets of unitary representations of  $G$ , then  $S$  is weakly contained in  $T$  ( $S \prec T$ ) if  $\bigcap_{\sigma \in S} \ker \sigma \supseteq \bigcap_{\tau \in T} \ker \tau$  or, equivalently, if every positive definite function associated to  $S$  can be uniformly approximated on compact subsets of  $G$  by sums of positive definite functions associated to  $T$ . Also  $S$  and  $T$  are weakly equivalent ( $S \sim T$ ) if  $S \prec T$  and  $T \prec S$ .

The dual space  $\widehat{G}$  of  $G$  is the set of equivalence classes of irreducible representations of  $G$ , endowed with the Jacobson topology. For a representation  $\pi$  of  $G$ , the support of  $\pi$  is the closed subset  $\text{supp } \pi = \{\rho \in \widehat{G} : \rho \prec \pi\}$  of  $\widehat{G}$ . In particular, the support of the left regular representation  $\lambda_G$  is the reduced dual  $\widehat{G}_r$ , and  $\lambda_G(C^*(G))$  is the so-called reduced group  $C^*$ -algebra of  $G$  which is denoted by  $C_r^*(G)$ . If  $N$  is a closed normal subgroup of  $G$ , then every representation of  $G/N$  can be lifted to a representation of  $G$ , and in this sense,  $(G/N)^\wedge$  will always be regarded as a subset of  $\widehat{G}$ . For general references to representation theory and dual spaces we mention [5] and [10].

$G$  is called amenable if there exists a non-zero positive linear functional  $m$  on the space  $C^b(G)$  of bounded continuous complex valued functions on  $G$  such that  $m({}_x f) = m(f)$  for all  $f \in C^b(G)$  and  $x \in G$ , where  ${}_x f(y) = f(x^{-1}y)$ . Recall that amenability is equivalent to a number of different conditions:  $C_r^*(G) = C^*(G)$ ,

$\widehat{G}_r = \widehat{G}$  or  $1_G \prec \lambda_G$  where  $1_G$  is the trivial one-dimensional representation of  $G$ . Concerning the theory of amenable groups, we refer the reader to [29] and [30].

Let  $H$  be a closed subgroup of  $G$ , and suppose that  $\sigma$  and  $\pi$  are representations of  $H$  and  $G$  respectively. The representation of  $G$  induced by  $\sigma$  is denoted  $\text{ind}_H^G \sigma$ . Then the tensor product  $\pi \otimes \text{ind}_H^G \sigma$  is equivalent to  $\text{ind}_H^G (\pi|_H \otimes \sigma)$ . Furthermore, by the theorem on induction in stages,  $\text{ind}_H^G \sigma = \text{ind}_K^G (\text{ind}_H^K \sigma)$  for every closed subgroup  $K$  of  $G$  containing  $H$ . Finally, we will frequently use that  $\pi \prec \text{ind}_H^G (\pi|_H)$  if  $G$  is amenable [12, Theorem 5.1].

Let  $A$  be a  $C^*$ -algebra and  $\widehat{A}$  its dual space, i.e. the set of equivalence classes of non-degenerated irreducible  $*$ -representations of  $A$ . We will several times use the fact that if the Banach space dual  $A^*$  of  $A$  is separable in the norm topology, then  $\widehat{A}$  is countable (see [15, Theorem 3.1] or [23, Lemma 4.12]).

## 2. WHEN IS $B_0(G)$ $w^*$ -CLOSED IN $B(G)$ ?

For a locally compact group  $G$ , let  $B_0(G) = B(G) \cap C_0(G)$  denote the norm-closed and translation invariant subalgebra of  $B(G)$  consisting of all functions in  $B(G)$  that vanish at infinity. In this section we shall study the problem of when  $B_0(G)$  is closed in  $B(G)$  with respect to the  $w^*$ -topology on  $B(G)$ .

It turns out to be appropriate to reformulate this condition in terms of convergence of positive definite functions. The following lemma, which is a consequence of [8, (2.1) and (2.12)], will be used frequently.

**Lemma 2.1.** *Let  $P(G)$  denote the set of all normalized continuous positive definite functions on  $G$ . The following are equivalent:*

- (i)  $B_0(G)$  is  $w^*$ -closed in  $B(G)$ .
- (ii) *If  $(\varphi_i)$  is a net in  $P(G) \cap C_0(G)$  converging to some  $\varphi \in P(G)$  uniformly on compact subsets of  $G$ , then  $\varphi \in C_0(G)$ .*

**Lemma 2.2.** *Suppose that  $B_0(G)$  is  $w^*$ -closed in  $B(G)$ . Then every amenable closed normal subgroup of  $G$  is compact. In particular, for an amenable group  $G$ ,  $B_0(G)$  is  $w^*$ -closed in  $B(G)$  only when  $G$  is compact.*

*Proof.* Let  $N$  be an amenable closed normal subgroup of  $G$ . Then the trivial one-dimensional representation  $1_N$  of  $N$  is weakly contained in the left regular representation  $\lambda_N$  of  $N$ . Hence, by continuity of inducing,  $\text{ind}_N^G 1_N$  is weakly contained in  $\text{ind}_N^G \lambda_N$ , which is equivalent to  $\lambda_G$ .

Let  $q : G \rightarrow G/N$  denote the quotient homomorphism. Then  $\text{ind}_N^G 1_N = \lambda_{G/N} \circ q$ . Thus every positive definite function  $\varphi$  associated to  $\lambda_{G/N} \circ q$  is a uniform on compacta limit of functions in  $A(G) \cap P(G) \subseteq B_0(G)$ . Hence, by hypothesis,  $\varphi \in C_0(G)$ . Since such a  $\varphi$  is constant on cosets of  $N$ ,  $N$  must be compact.  $\square$

We continue with two inheritance properties.

**Lemma 2.3.** *Let  $H$  be an open subgroup of the locally compact group  $G$ . If  $B_0(G)$  is  $w^*$ -closed in  $B(G)$ , then  $B_0(H)$  is  $w^*$ -closed in  $B(H)$ .*

*Proof.* Let  $(\varphi_i)$  be a net in  $P(H) \cap C_0(H)$  such that  $\varphi_i \rightarrow \varphi \in P(H)$  uniformly on compact subsets of  $H$ . Let  $\widetilde{\varphi_i}$  and  $\widetilde{\varphi}$  denote the trivial extensions of  $\varphi_i$  and  $\varphi$  to  $G$ , that is,  $\widetilde{\varphi}(x) = \varphi_i(x) = 0$  for  $x \in G \setminus H$ . Clearly, then  $\widetilde{\varphi_i} \in P(G) \cap C_0(G)$ ,  $\widetilde{\varphi} \in P(G)$  and  $\widetilde{\varphi_i} \rightarrow \widetilde{\varphi}$  uniformly on compact subsets of  $G$ . By hypothesis,  $\widetilde{\varphi} \in C_0(G)$  and hence  $\varphi \in C_0(H)$ .  $\square$

**Lemma 2.4.** *Let  $G$  be a locally compact group and  $K$  a compact normal subgroup of  $G$ . Then  $B_0(G)$  is  $w^*$ -closed in  $B(G)$  if and only if  $B_0(G/K)$  is  $w^*$ -closed in  $B(G/K)$ .*

*Proof.* Suppose that  $B_0(G)$  is  $w^*$ -closed in  $B(G)$ , and let  $(\varphi_\iota)$  be a net in  $P(G/K) \cap C_0(G/K)$  converging to some  $\varphi \in P(G/K)$  uniformly on compact subsets of  $G/K$ . Then, with  $q : G \rightarrow G/K$  the quotient homomorphism,  $\varphi_\iota \circ q \rightarrow \varphi \circ q$  uniformly on compact subsets of  $G$  and  $\varphi_\iota \circ q \in C_0(G)$  since  $K$  is compact. Hence, by hypothesis,  $\varphi \circ q \in C_0(G)$  and so  $\varphi \in C_0(G/K)$ .

Conversely, suppose that  $B_0(G/K)$  is  $w^*$ -closed in  $B(G/K)$ , and let  $\varphi \in P(G)$  and  $(\varphi_\iota) \subseteq P(G) \cap C_0(G)$  such that  $\varphi_\iota \rightarrow \varphi$  uniformly on compact subsets of  $G$ . Define  $\psi_\iota$  and  $\psi$  on  $G/K$  by

$$\psi_\iota(xK) = \int_K |\varphi_\iota(xk)|^2 dk \text{ and } \psi(xK) = \int_K |\varphi(xk)|^2 dk,$$

$x \in G$  ( $dk$  being the normalized Haar measure on  $K$ ). Then

$$\psi \in P(G/K) \text{ and } \psi_\iota \in P(G/K) \cap C_0(G/K),$$

and  $\psi_\iota \rightarrow \psi$  uniformly on compact subsets of  $G/K$ . Hence  $\psi \in C_0(G/K)$ .

For  $\delta \in \widehat{K}$ , let  $\chi_\delta$  denote the corresponding minimal idempotent in  $L^1(K)$ . Then by the Cauchy-Schwarz inequality,

$$|\varphi * \chi_\delta(x)| \leq |\psi(xK)|^{1/2}$$

for every  $x \in G$ . Since  $K$  is compact, this implies  $\varphi * \chi_\delta \in C_0(G)$  for each  $\delta \in \widehat{K}$ . Now the linear span of  $\{\chi_\delta : \delta \in \widehat{K}\}$  is dense in  $Z(L^1(K))$ , the centre of  $L^1(K)$ . It follows that for any  $f \in Z(L^1(K))$ ,  $\varphi * f$  is a uniform limit on  $G$  of finite linear combinations of functions  $\varphi * \chi_\delta$ ,  $\delta \in \widehat{K}$ . Hence  $\varphi * f \in C_0(G)$  for every  $f \in Z(L^1(K))$ . Finally, taking for  $f$  functions in  $Z(L^1(K))$  with support shrinking to  $\{e\}$ , we easily conclude that  $\varphi \in C_0(G)$ . This completes the proof.  $\square$

We now turn to connected Lie groups. Theorem 2.7 below is the key result in this section.

**Lemma 2.5.** *Let  $G$  be a connected Lie group and  $N$  a connected closed normal subgroup. If  $B_0(G)$  is  $w^*$ -closed in  $B(G)$ , then the centre  $Z(N)$  of  $N$  is compact and  $N/Z(N)$  is semisimple.*

*Proof.* Let  $R$  denote the radical of  $N$ . Then  $R$  and  $Z(N)$  are amenable normal subgroups of  $G$ , and therefore both must be compact by Lemma 2.2. Since  $R$  is solvable and connected Lie, it is isomorphic to a torus  $\mathbb{T}^n$ . Hence  $\text{Aut}(R)$ , the automorphism group of  $R$ , is discrete. Now,  $G$  acts by conjugation on  $R$ , and this defines a continuous homomorphism from  $G$  into  $\text{Aut}(R)$ .  $G$  being connected, this homomorphism has to be trivial. This shows that  $R$  is contained in the centre of  $G$ . So  $R \subseteq Z(N)$  and hence  $N/Z(N)$  is semisimple.  $\square$

We remind the reader that a locally compact group  $G$  is said to have Kazhdan's property (T) if the trivial representation  $1_G$  is an isolated point in the dual space  $\widehat{G}$ . An amenable group satisfies (T) if and only if it is compact. On the other hand, many connected semisimple Lie groups and many discrete groups share property (T). A comprehensive account on groups with property (T) has been given in [14].

**Lemma 2.6.** *Let  $G$  be a connected Lie group such that  $B_0(G)$  is  $w^*$ -closed in  $B(G)$ . Then  $G$  has property (T).*

*Proof.* By Lemma 2.5,  $G$  is reductive with compact centre. Let

$$\tilde{G} = \mathbb{R}^n \times G_1 \times \cdots \times G_m$$

be the universal covering group of  $G$ , where  $G_1, \dots, G_m$  are simply connected Lie groups. Denote by  $Z_i$  the (discrete) centre of  $G_i$ ,  $i = 1, \dots, m$ . Then  $G = \tilde{G}/\Gamma$  for some discrete subgroup  $\Gamma$  of the centre  $Z(\tilde{G}) = \mathbb{R}^n \times Z_1 \times \cdots \times Z_m$  of  $\tilde{G}$ . Moreover,  $Z(G) = Z(\tilde{G})/\Gamma$ . Hence

$$G/Z(G) = \tilde{G}/Z(\tilde{G}) = G_1/Z_1 \times \cdots \times G_m/Z_m.$$

Since  $Z(G)$  is compact,  $B_0(G/Z(G))$  is  $w^*$ -closed in  $B(G/Z(G))$  (Lemma 2.4). Assume, towards a contradiction, that  $G$  does not have property (T). Then  $G/Z(G)$  does not have property (T). Hence some factor, say  $G_1/Z_1$ , fails to have property (T) (see [35, Lemma 7.4.1]). Now, recall the following result due to Howe and Moore [35, Theorem 2.2.20]. If  $\pi$  is a unitary representation of a simple Lie group with finite centre and if there are no non-zero  $\pi$ -invariant vectors, then all the matrix coefficients of  $\pi$  vanish at infinity. Therefore, there exists a sequence

$$(\varphi_n^{(1)}) \subseteq P(G_1/Z_1) \cap C_0(G_1/Z_1)$$

converging to 1 uniformly on compact subsets of  $G_1/Z_1$ . Observe that  $G_1/Z_1$  is not compact. Now choose arbitrary

$$\varphi^{(k)} \in P(G_k/Z_k) \cap C_0(G_k/Z_k),$$

$k = 2, \dots, m$ , and set

$$\varphi_n = \varphi_n^{(1)} \times \varphi^{(2)} \times \cdots \times \varphi^{(m)}.$$

Clearly,  $(\varphi_n)$  is a sequence of continuous positive definite functions on  $G/Z(G)$  that vanish at infinity, and

$$\varphi_n \rightarrow 1 \times \varphi^{(2)} \times \cdots \times \varphi^{(m)}$$

uniformly on compact subsets of  $G/Z(G)$ . Since  $1 \times \varphi^{(2)} \times \cdots \times \varphi^{(m)}$  does not vanish at infinity, we have reached a contradiction. Thus  $G$  has property (T).  $\square$

**Theorem 2.7.** *Let  $G$  be a connected Lie group. Then  $B_0(G)$  is  $w^*$ -closed in  $B(G)$  if and only if  $G$  is a reductive Lie group with compact centre and Kazhdan's property (T).*

*Proof.* From Lemma 2.5, applied to  $N = G$ , and Lemma 2.6 we know that  $w^*$ -closedness of  $B_0(G)$  in  $B(G)$  implies the stated conditions on  $G$ .

Suppose now that  $G$  is a connected reductive Lie group with compact centre  $Z(G)$  and property (T). According to Lemma 2.4 it suffices to show that  $B_0(G/Z(G))$  is  $w^*$ -closed in  $B(G/Z(G))$ . Recall from the proof of Lemma 2.6 that  $G/Z(G)$  has a decomposition

$$G/Z(G) = G_1/Z_1 \times \cdots \times G_m/Z_m,$$

where  $G_1, \dots, G_m$  are connected simple Lie groups with centres  $Z_1, \dots, Z_m$ , respectively. Set  $H = G/Z(G)$  and  $H_i = G_i/Z_i$  for  $i = 1, \dots, m$ , and observe that every  $H_i$  has property (T). Of course, according to Lemma 2.4 we can assume that none of the  $H_i$  is compact.

Let  $\varphi_n \in P(H) \cap C_0(H)$ ,  $n \in \mathbb{N}$ , such that  $\varphi_n \rightarrow \varphi$  uniformly on compact subsets of  $G$  for some  $\varphi \in P(H)$ . Let  $\pi_n$  and  $\pi$  denote the representations of  $H$  associated to  $\varphi_n$  and  $\varphi$  through the GNS-construction. Then  $\pi$  is weakly contained in the direct sum  $\bigoplus_{n=1}^{\infty} \pi_n$ .

We claim that the restriction  $\pi|_{H_i}$  of  $\pi$  to  $H_i$  does not contain the trivial representation  $1_{H_i}$ . Indeed, otherwise for some  $n \in \mathbb{N}$ ,  $\pi_n|_{H_i}$  contains  $1_{H_i}$  since  $H_i$  has property (T). However, since  $\varphi_n \in C_0(H)$ , all the matrix coefficients of  $\pi_n$  vanish at infinity.

We have thus verified that  $H$  satisfies the hypotheses of the Howe-Moore theorem [35, Theorem 2.2.20]. It follows that all the matrix coefficients of  $\pi$  vanish at infinity. This proves that  $\varphi \in C_0(H)$ .  $\square$

In order to deal with almost connected groups we need one more lemma.

**Lemma 2.8.** *Let  $G$  be a locally compact group and  $H$  a closed subgroup such that  $G/H$  is compact. If  $B_0(H)$  is  $w^*$ -closed in  $B(H)$ , then  $B_0(G)$  is  $w^*$ -closed in  $B(G)$ .*

*Proof.* For any function  $\phi$  on  $G$  and  $x, y \in G$ , let

$${}_x\phi(y) = \phi(x^{-1}y), \phi_x(y) = \phi(yx) \text{ and } \phi^x(y) = \phi(x^{-1}yx).$$

Notice first that if  $\psi$  is a positive definite function on  $G$ , then  $\psi^x + \psi + {}_x\psi + \psi_x$  and  $\psi^x + \psi + i({}_x\psi - \psi_x)$  are also positive definite for every  $x \in G$ . Indeed, for all  $f \in L^1(G)$

$$\langle \psi^x + \psi + {}_x\psi + \psi_x, f \rangle = \langle \psi, (\delta_x + \delta_e)^* * f * (\delta_x + \delta_e) \rangle$$

and

$$\langle \psi^x + \psi + i({}_x\psi - \psi_x), f \rangle = \langle \psi, (\delta_e - i\delta_x)^* * f * (\delta_e - i\delta_x) \rangle.$$

Let  $(\varphi_\iota)$  be a net in  $P(G) \cap C_0(G)$  converging to some  $\varphi \in P(G)$  uniformly on compact subsets of  $G$ . Then, uniformly on compact subsets of  $G$ ,  $\varphi_\iota^x \rightarrow \varphi^x$ ,

$$\varphi_\iota^x + \varphi_\iota + {}_x\varphi_\iota + (\varphi_\iota)_x \rightarrow \varphi^x + \varphi + {}_x\varphi + \varphi_x$$

and

$$\varphi_\iota^x + \varphi_\iota + i({}_x\varphi_\iota - (\varphi_\iota)_x) \rightarrow \varphi^x + \varphi + i({}_x\varphi - \varphi_x)$$

for every  $x \in G$ . Thus, since  $B_0(H)$  is  $w^*$ -closed in  $B(H)$ ,

$$\varphi|_H, \varphi^x|_H, (\varphi^x + \varphi + {}_x\varphi + \varphi_x)|_H \text{ and } (\varphi^x + \varphi + i({}_x\varphi - \varphi_x))|_H$$

vanish at infinity on  $H$ . It follows that  ${}_x\varphi|_H \in C_0(H)$  for each  $x \in G$ . Since  $G/H$  is compact, employing the uniform continuity of  $\varphi$ , it is easily verified that  $\varphi \in C_0(G)$ .  $\square$

The converse to Lemma 2.8 does not hold in general. That is, if  $B_0(G)$  is  $w^*$ -closed in  $B(G)$  and  $H$  is a closed cocompact subgroup of  $G$ , then  $B_0(H)$  need not be  $w^*$ -closed in  $B(H)$ . As an example, take for  $G$  a simply connected Lie group with finite centre and property (T) and for  $H$  a minimal parabolic subgroup. Then by the Howe-Moore result referred to in the proofs of Lemma 2.6 and Theorem 2.7,  $B_0(G)$  is  $w^*$ -closed in  $B(G)$ , while  $B_0(H)$  fails to be  $w^*$ -closed in  $B(H)$  since  $H$  is non-compact and amenable.

**Corollary 2.9.** *Let  $G$  be a connected Lie group and  $N$  a connected closed normal subgroup of  $G$  such that  $G/N$  is compact. Then  $B_0(G)$  is  $w^*$ -closed in  $B(G)$  if and only if  $B_0(N)$  is  $w^*$ -closed in  $B(N)$ .*

*Proof.* Suppose that  $B_0(G)$  is  $w^*$ -closed in  $B(G)$ . By Lemma 2.5,  $Z(N)$  is compact and  $N/Z(N)$  is semisimple. Also, since  $G$  has property (T) by Lemma 2.6 and  $G/N$  is compact,  $N$  has property (T) [34, Theorem 3.7]. By Theorem 2.7 this implies that  $B_0(N)$  is  $w^*$ -closed in  $B(N)$ .

The converse is a special case of Lemma 2.8.  $\square$

**Theorem 2.10.** *Let  $G$  be an almost connected locally compact group. Then  $B_0(G)$  is  $w^*$ -closed in  $B(G)$  if and only if the connected component  $G_0$  of  $G$  is a projective limit of reductive Lie groups with property (T) and compact centres.*

*Proof.* Suppose first that  $G_0$  has the indicated structure. Choose a compact normal subgroup  $K$  of  $G_0$  such that  $G_0/K$  is a reductive Lie group with property (T) and compact centre. By Theorem 2.7,  $B_0(G_0/K)$  is  $w^*$ -closed in  $B(G_0/K)$ . Since  $K$  and  $G/G_0$  are compact, an application of Lemmas 2.4 and 2.8 yields that  $B_0(G)$  is  $w^*$ -closed in  $B(G)$ .

Conversely, suppose that  $B_0(G)$  is  $w^*$ -closed in  $B(G)$ .  $G$  being almost connected it is a projective limit of Lie groups  $G/K_\iota$ . Thus there are closed normal subgroups  $H_\iota$  of finite index in  $G$  such that  $K_\iota \subseteq H_\iota$  and  $H_\iota/K_\iota = (G/K_\iota)_0$ . Then  $G_0$  is the projective limit of the groups  $G_0/G_0 \cap K_\iota$ , and the  $G_0/G_0 \cap K_\iota$  are connected Lie groups since

$$G_0/G_0 \cap K_\iota = G_0 K_\iota / K_\iota,$$

a closed connected subgroup of  $G/K_\iota$ . By Theorem 2.7 it suffices to show that  $B_0(G_0/G_0 \cap K_\iota)$  is  $w^*$ -closed in  $B(G_0/G_0 \cap K_\iota)$ .

Now, since  $B_0(G)$  is  $w^*$ -closed in  $B(G)$ ,  $B_0(G/K_\iota)$  is  $w^*$ -closed in  $B(G/K_\iota)$  by Lemma 2.4, and hence  $B_0(H_\iota/K_\iota)$  is  $w^*$ -closed in  $B(H_\iota/K_\iota)$  by Lemma 2.3. Moreover,  $G_0 K_\iota / K_\iota$  is a cocompact connected normal subgroup of the connected Lie group  $H_\iota / K_\iota$ . Thus, by Corollary 2.9,  $B_0(G_0 K_\iota / K_\iota)$  is  $w^*$ -closed in  $B(G_0 K_\iota / K_\iota)$ . This proves that  $B_0(G_0/G_0 \cap K_\iota)$  is  $w^*$ -closed in  $B(G_0/G_0 \cap K_\iota)$ .  $\square$

We conclude this section with some remarks.

*Remarks 2.11.* (i) The connected reductive Lie groups with property (T) and compact centres are precisely the groups of the form  $G = (\mathbb{R}^n \times G_1 \times \cdots \times G_m)/\Gamma$ , where  $G_1, \dots, G_m$  are simple Lie groups not locally isomorphic to  $\mathrm{SO}(k, 1)$ ,  $k \geq 2$ , or  $\mathrm{SU}(k, 1)$ ,  $k \geq 1$ , and  $\Gamma$  is a discrete cocompact subgroup of  $\mathbb{R}^n \times Z_1 \times \cdots \times Z_m$ , the centre of  $\mathbb{R}^n \times G_1 \times \cdots \times G_m$ .

Indeed, if  $G$  is of this form, then  $G/Z(G) = G_1 \times \cdots \times G_m$ , where  $Z(G)$  denotes the centre of  $G$ , has property (T) (see [14, Chap. 2, 13. Corollaire, 9. Remarque and Chap. 9]). As  $Z(G)$  is compact,  $G$  has property (T) [14, Chap. 1, 9. Proposition].

Conversely, let  $G$  be a connected reductive Lie group with property (T) and compact centre. Let  $\tilde{G} = \mathbb{R}^n \times G_1 \times \cdots \times G_m$  be its universal covering group, where  $G_1, \dots, G_m$  are simple Lie groups with centres  $Z_1, \dots, Z_m$ . The arguments used in the proof of Lemma 2.6 show that  $G_1, \dots, G_m$  have property (T) and hence are not locally isomorphic to  $\mathrm{SO}(k, 1)$  or  $\mathrm{SU}(k, 1)$  (see [14, Chap. 6, 23. Corollaire]).

(ii) Let  $G$  be a discrete group such that  $B_0(G)$  is  $w^*$ -closed in  $B(G)$ . Then every element in  $G$  has finite order. Indeed, this follows immediately by applying Lemma 2.2 to the cyclic subgroups of  $G$ .

(iii) Let  $G$  be a linear group (that is, a subgroup  $\mathrm{GL}(n, K)$  for some field  $K$ ) with the discrete topology. If  $G$  is infinite, then  $B_0(G)$  is not  $w^*$ -closed in  $B(G)$ . In fact, this is clear from (ii) if  $G$  has an element of infinite order. On the other

hand, if  $G$  is a torsion group, then it is well known to be locally finite [6, Theorem 9.2] and hence amenable, so that the claim follows from Lemma 2.2.

### 3. CHARACTERIZATIONS OF COMPACT GROUPS

If  $G$  is a compact group then  $A(G) = B(G)$ . It seems likely that the converse is also true; i.e.  $w^*$ -closedness of  $A(G)$  in  $B(G)$  already forces  $G$  to be compact. We have been able to show this for groups containing an almost connected open normal subgroup (Theorem 3.6). The case that remains open is that of a totally disconnected group.

We start with a lemma which will be generalized in Section 4 (Lemma 4.2).

**Lemma 3.1.** *If  $A(G)$  is  $w^*$ -closed in  $B(G)$  and  $H$  is an open subgroup of  $G$ , then  $A(H)$  is  $w^*$ -closed in  $B(H)$ .*

*Proof.* It suffices to show that the unit ball of  $A(H)$  is  $w^*$ -closed in the unit ball of  $B(H)$ . Thus, let  $(\varphi_\iota)$  be a net in  $A(H)$  and  $\varphi \in B(H)$  such that

$$\|\varphi_\iota\| \leq 1, \|\varphi\| \leq 1 \text{ and } \varphi_\iota \rightarrow \varphi$$

in the  $w^*$ -topology. Let  $\tilde{\varphi}_\iota$  and  $\tilde{\varphi}$  denote the trivial extensions of  $\varphi_\iota$  and  $\varphi$  to  $G$ . Then  $\tilde{\varphi}_\iota \in A(G)$ ,  $\|\tilde{\varphi}_\iota\| \leq 1$  and, for each  $f \in L^1(G)$ ,

$$\begin{aligned} \int_G \tilde{\varphi}_\iota(x) f(x) dx &= \int_H \varphi_\iota(h) f(h) dh \\ &\rightarrow \int_H \varphi(h) f(h) dh = \int_G \tilde{\varphi}(x) f(x) dx. \end{aligned}$$

Hence  $\tilde{\varphi}_\iota \rightarrow \tilde{\varphi}$  in the  $\sigma(B(G), L^1(G))$ -topology. Since  $(\tilde{\varphi}_\iota)$  is a bounded net, it follows that  $(\tilde{\varphi}_\iota)$  is  $w^*$ -convergent to  $\tilde{\varphi}$ . By hypothesis,  $\tilde{\varphi} \in A(G)$  and so  $\varphi \in A(H)$ .  $\square$

**Lemma 3.2.** *Let  $K$  be a compact normal subgroup of  $G$ . If  $A(G)$  is  $w^*$ -closed in  $B(G)$ , then  $A(G/K)$  is  $w^*$ -closed in  $B(G/K)$ .*

*Proof.* Consider the map  $T_K : f \rightarrow T_K f$  from  $L^1(G)$  onto  $L^1(G/K)$  given by

$$T_K f(xK) = \int_K f(xk) dk.$$

This map extends to a  $*$ -homomorphism from  $C^*(G)$  onto  $C^*(G/K)$  with dual map  $T_K^* : B(G/K) \rightarrow B(G)$ . Furthermore,  $T_K^*(B(G/K))$  consists precisely of those functions in  $B(G)$  that are constant on cosets of  $K$  [8, (2.26)]. Also, since  $K$  is compact,

$$T_K^*(A(G/K)) = A(G) \cap T_K^*(B(G/K)).$$

Now, let  $\varphi_\iota \in A(G/K)$  such that  $\varphi_\iota \rightarrow \varphi$  in the  $w^*$ -topology for some  $\varphi \in B(G/K)$ . Then

$$\langle T_K^*(\varphi_\iota), f \rangle = \langle \varphi_\iota, T_K(f) \rangle \rightarrow \langle \varphi, T_K(f) \rangle = \langle T_K^*(\varphi), f \rangle$$

for each  $f \in C^*(G)$ . Thus, by hypothesis,  $T_K^*(\varphi) \in A(G)$  and so

$$T_K^*(\varphi) \in A(G) \cap T_K^*(B(G/K)),$$

whence  $\varphi \in A(G/K)$ .  $\square$

**Lemma 3.3.** *Let  $G$  be an almost connected locally compact group. If  $A(G)$  is  $w^*$ -closed in  $B(G)$ , then  $G$  is compact.*

*Proof.* Since an almost connected group is a projective limit of Lie groups and  $A(G/K)$  is  $w^*$ -closed in  $B(G/K)$  for every compact normal subgroup  $K$  of  $G$  (Lemma 3.2), we can assume that  $G$  is a Lie group. Being a compactly generated Lie group,  $G$  is second countable and hence  $A(G)$  is a separable Banach space. By hypothesis,

$$A(G) = B_\lambda(G) = C_r^*(G)^*.$$

Now, a  $C^*$ -algebra  $A$  with separable dual Banach space has a countable dual  $\widehat{A}$  (see Section 1). It follows that  $\widehat{G}_r$ , the reduced dual of  $G$ , is countable. Finally, by [2, Theorem 2.5] a separable Lie group with countable reduced dual is compact. This shows that  $G$  is compact.  $\square$

**Corollary 3.4.** *Let  $G$  be any locally compact group and suppose that  $A(G)$  is  $w^*$ -closed in  $B(G)$ . Then  $G$  contains a compact open subgroup.*

*Proof.* Since  $G/G_0$  is totally disconnected, there exists an open subgroup  $H$  of  $G$  so that  $H/G_0$  is compact. By Lemma 3.1,  $A(H)$  is  $w^*$ -closed and hence  $H$  is compact by Lemma 3.3.  $\square$

**Lemma 3.5.** *If  $G$  is a discrete group and  $A(G)$  is  $w^*$ -closed in  $B(G)$ , then  $G$  is finite.*

*Proof.* Assume that  $G$  is infinite. Then  $G$  has a countable infinite subgroup  $H$ . By Lemma 3.1,  $A(H)$  is  $w^*$ -closed in  $B(H)$ . As in the proof of Lemma 3.3 we now conclude that  $\widehat{H}_r$  is countable. Applying Baggett's result again, it follows that  $H$  is finite, a contradiction.  $\square$

**Theorem 3.6.** *Suppose that  $G$  contains an almost connected open normal subgroup. Then  $A(G)$  is  $w^*$ -closed in  $B(G)$  if and only if  $G$  is compact.*

*Proof.* Let  $N$  be an almost connected open normal subgroup of  $G$ . Then  $A(N)$  is  $w^*$ -closed in  $B(N)$  by Lemma 3.1, and Lemma 3.3 implies that  $N$  is compact. By Lemma 3.2,  $A(G/N)$  is  $w^*$ -closed in  $B(G/N)$ . Since  $N$  is open, Lemma 3.5 gives that  $G/N$  is finite. Thus  $G$  is compact.  $\square$

We now turn to a second characterization of compact groups in terms of certain properties of the  $w^*$ -topology on  $B(G)$ . If  $G$  is a compact group, then the  $w^*$ -topology and the norm topology agree on the unit sphere of  $B(G) = A(G)$  [11, Corollary 2]. We are going to establish the converse to this (see [22, Theorem 5] for the amenable case). Actually, we prove a stronger result in that we replace the unit sphere of  $B(G)$  by the smaller set  $P_\lambda(G) = B_\lambda(G) \cap P(G)$  of all normalized positive definite functions on  $G$  associated to representations that are weakly contained in the left regular representation. Note that this property implies the Radon-Nikodym property for  $B(G)$  but not conversely (see [11] and [33]).

For any locally compact group  $G$ , let  $P_\lambda(G) = B_\lambda(G) \cap P(G)$ , the set of all normalized continuous positive definite functions on  $G$  associated to representations that are weakly contained in the left regular representation.  $P_\lambda(G)$  is a  $w^*$ -compact convex subset of  $B(G)$ . We denote by  $\text{ex}(P_\lambda(G))$  the set of extreme points of the  $w^*$ -compact convex subset  $P_\lambda(G)$  of  $B(G)$ .

**Lemma 3.7.** *Let  $G$  be a locally compact group and  $\varphi \in \text{ex}(P_\lambda(G))$ . Suppose that  $\varphi$  is a point of continuity of the identity map*

$$(\text{ex}(P_\lambda(G)), w^*) \rightarrow (\text{ex}(P_\lambda(G)), \|\cdot\|).$$

*Then  $\pi_\varphi$  is an isolated point in  $\widehat{G}_r$ .*

*Proof.* Notice first that  $\text{ex}(P_\lambda(G)) \subseteq \text{ex}(P(G))$  because if  $\varphi \in P_\lambda(G)$  and  $\psi \in P(G)$  are such that  $c\varphi - \psi$  is positive definite for some  $c \geq 0$ , then  $\psi \in P_\lambda(G)$ . By [5, 2.12.1], if  $\varphi_1, \varphi_2 \in \text{ex}(P(G))$  and  $\pi_{\varphi_1}$  and  $\pi_{\varphi_2}$  are not equivalent, then  $\|\varphi_1 - \varphi_2\| \geq 2$ . By assumption there exists a  $w^*$ -open subset  $U$  of  $\text{ex}(P_\lambda(G))$  such that

$$U \subseteq \{\psi \in \text{ex}(P_\lambda(G)) : \|\psi - \varphi\| < 2\}.$$

It follows that  $\pi_\psi = \pi_\varphi$  for all  $\psi \in U$ . Now, by [5, Theorem 3.4.11], the map  $q : \psi \rightarrow \pi_\psi$  from  $\text{ex}(P_\lambda(G))$  onto  $\widehat{G}_r$  is open. Thus  $\{\pi_\varphi\} = q(U)$  is open in  $\widehat{G}_r$ .  $\square$

**Lemma 3.8.** *Let  $H$  be an open subgroup of  $G$ . If the identity map from  $(P_\lambda(G), w^*)$  to  $(P_\lambda(H), \|\cdot\|)$  is continuous, then the identity map from  $(P_\lambda(H), w^*)$  to  $(P_\lambda(H), \|\cdot\|)$  is continuous.*

*Proof.* For any  $\varphi \in P_\lambda(H)$ , the trivial extension  $\tilde{\varphi}$  belongs to  $P_\lambda(G)$ . Indeed,  $\tilde{\varphi}$  is a positive definite function associated to the induced representation  $\text{ind}_H^G \pi_\varphi$ , and  $\pi_\varphi \prec \lambda_H$  implies

$$\text{ind}_H^G \pi_\varphi \prec \text{ind}_H^G \lambda_H = \lambda_G.$$

Let  $(\varphi_\alpha)$  be a net in  $P_\lambda(H)$  converging to  $\varphi \in P_\lambda(H)$  in the  $w^*$ -topology. Then  $\tilde{\varphi}_\alpha \rightarrow \tilde{\varphi}$  in the  $w^*$ -topology on  $P_\lambda(G)$  (compare the proof of Lemma 3.1). By hypothesis,  $\|\tilde{\varphi}_\alpha - \tilde{\varphi}\| \rightarrow 0$  and hence  $\|\varphi_\alpha - \varphi\| \rightarrow 0$ .  $\square$

**Theorem 3.9.** *For any locally compact group  $G$  the following conditions are equivalent.*

- (i)  $G$  is compact.
- (ii) The  $w^*$ -topology and the norm topology agree on the unit sphere of  $B(G)$ .
- (iii) The  $w^*$ -topology and the norm topology agree on  $P_\lambda(G)$ .

*Proof.* As mentioned above, (i) $\Rightarrow$ (ii) is due to Granirer and Leinert [11]. Since (ii) $\Rightarrow$ (iii) is trivial, it only remains to prove (iii) $\Rightarrow$ (i).

Assume that  $G$  fails to be compact. Then  $G$  contains a non-compact,  $\sigma$ -compact, open subgroup  $H$ . By Lemma 3.8, the  $w^*$ -topology and the norm topology coincide on  $P_\lambda(H)$ . It follows from Lemma 3.7 that  $\widehat{H}_r$  is discrete. Since  $H$  is  $\sigma$ -compact Theorem 7.6 of [34] now shows that  $H$  is compact, a contradiction.  $\square$

#### 4. WHEN IS $A_\gamma(G)$ $w^*$ -CLOSED IN $B(G)$ ?

For a locally compact group  $G$  and any unitary representation  $\pi$  of  $G$ , the Fourier space  $A_\pi(G)$  associated to  $\pi$  is defined to be the norm-closed linear subspace of  $B(G)$  generated by all the coordinate functions of  $\pi$  [1], that is, the functions of the form  $x \rightarrow \langle \pi(x)\xi, \eta \rangle$ ,  $\xi, \eta \in \mathcal{H}_\pi$ .

The conjugation representation  $\gamma_G$  (or simply  $\gamma$ , if no confusion can arise) on  $L^2(G)$  is defined by

$$\gamma_G(x)f(y) = \Delta(x)^{1/2}f(x^{-1}yx),$$

$f \in L^2(G)$ ,  $x, y \in G$ . The purpose of this section is to investigate the question of when  $A_\gamma(G)$  is  $w^*$ -closed in  $B(G)$ . It will turn out that this is closely related to problems on the support of  $\gamma$  as studied in [21]. We start with two simple facts on  $A_\pi(G)$  for general representations  $\pi$ .

**Lemma 4.1.** *If  $G$  is a compact group, then  $A_\pi(G)$  is  $w^*$ -closed in  $B(G)$  for every representation  $\pi$  of  $G$ .*

*Proof.* The  $w^*$ -closure  $\overline{A_\pi(G)}^{w^*}$  of  $A_\pi(G)$  is the dual space of the  $C^*$ -algebra  $\pi(C^*(G))$ , which is a quotient of  $C^*(G)$ . Hence each  $\varphi \in \overline{A_\pi(G)}^{w^*}$  is a linear combination of positive definite functions in  $\overline{A_\pi(G)}^{w^*}$ . Therefore, it suffices to prove that every positive definite  $\varphi \in \overline{A_\pi(G)}^{w^*}$  actually is in  $A_\pi(G)$ .

For that, notice that there is a net  $(\varphi_\alpha)$  in  $A_\pi(G)$  such that  $\varphi_\alpha \rightarrow \varphi$  in the  $w^*$ -topology and  $\|\varphi_\alpha\| \rightarrow \|\varphi\|$  (compare [10, p. 565]). By [11, Theorem A] it follows that  $\|\varphi_\alpha\psi - \varphi\psi\| \rightarrow 0$  for every  $\psi \in A(G)$ . In particular,  $\|\varphi_\alpha - \varphi\| \rightarrow 0$  by setting  $\psi = 1 \in B(G) = A(G)$ . This shows that  $\varphi \in A_\pi(G)$ .  $\square$

**Lemma 4.2.** *Suppose that  $\pi$  is a representation of  $G$  such that  $A_\pi(G)$  is  $w^*$ -closed in  $B(G)$ . Then  $A_{\pi|_H}(H)$  is  $w^*$ -closed in  $B(H)$  for every open subgroup  $H$  of  $G$ .*

*Proof.* Recall that, by [1, Theorem 2.2],  $\varphi \in B(G)$  belongs to  $A_\pi(G)$  if and only if  $\varphi$  can be written as

$$\varphi = \sum_{n=1}^{\infty} \langle \pi(\cdot) \xi_n, \eta_n \rangle$$

where  $\xi_n, \eta_n \in \mathcal{H}_\pi$  and  $\sum_{n=1}^{\infty} \|\xi_n\| \cdot \|\eta_n\| < \infty$ . In particular,  $A_{\pi|_H}(H) = A_\pi(G)|_H$ . It suffices to show that the unit ball of  $A_{\pi|_H}(H)$  is  $w^*$ -closed in the unit ball of  $B(H)$ . Thus, let  $\varphi_i \in A_{\pi|_H}(H)$ ,  $i \in I$ , and  $\varphi \in B(H)$  such that  $\|\varphi_i\| \leq 1$ ,  $\|\varphi\| \leq 1$  and  $\varphi_i \rightarrow \varphi$  in the  $w^*$ -topology. Choose representations

$$\varphi_i = \sum_{n=1}^{\infty} \langle \pi(\cdot) \xi_{in}, \eta_{in} \rangle$$

such that  $\sum_{n=1}^{\infty} \|\xi_{in}\| \cdot \|\eta_{in}\| \leq 2$  (see [1, Proposition 2.9]). Define

$$\psi_i(x) = \sum_{n=1}^{\infty} \langle \pi(x) \xi_{in}, \eta_{in} \rangle$$

for all  $x \in G$  and  $i \in I$ . Then  $\psi_i \in A_\pi(G)$  and  $\|\psi_i\| \leq 2$ . Since the unit ball in  $B(G)$  is  $w^*$ -compact, we can assume that  $\psi_i \rightarrow \psi$  in the  $w^*$ -topology for some  $\psi \in B(G)$ . Now,  $A_\pi(G)$  is  $w^*$ -closed in  $B(G)$ , so that  $\psi \in A_\pi(G)$  and hence  $\psi|_H \in A_{\pi|_H}(H)$ . On the other hand, the restriction map  $B(G) \rightarrow B(H)$  is  $w^*$ -continuous as  $H$  is open (see Section 5). Indeed, this follows from the fact that  $C^*(H)$  is a subalgebra of  $C^*(G)$  whenever  $H$  is open in  $G$ . Thus

$$\varphi_i = \psi_i|_H \rightarrow \psi|_H \text{ and } \varphi_i \rightarrow \varphi$$

in the  $w^*$ -topology. This proves  $\varphi = \psi|_H \in A_{\pi|_H}(H)$ .  $\square$

We now apply the preceding lemmas to the conjugation representation. The following corollary will be used several times in the sequel.

**Corollary 4.3.** *Suppose that  $G$  is second countable and  $A_{\gamma_G}(G)$  is  $w^*$ -closed in  $B(G)$ . Then, for every open subgroup  $H$  of  $G$ ,  $\text{supp } \gamma_H$  is countable.*

*Proof.* Since  $G$  is second countable,  $A_\gamma(G)$  is norm separable. Since the restriction map from  $B(G)$  to  $B(H)$  is norm continuous,  $A_{\gamma_G|_H}(H) = A_{\gamma_G}(G)|_H$  is norm

separable. Now, since  $\gamma_H$  is a subrepresentation of  $\gamma_G|_H$  and  $A_{\gamma_G|_H}(H)$  is  $w^*$ -closed in  $B(H)$  by Lemma 4.2,

$$\overline{A_{\gamma_H}(H)}^{w^*} \subseteq A_{\gamma_G|_H}(H),$$

so that  $\overline{A_{\gamma_H}(H)}^{w^*}$  is norm separable. Thus  $\gamma_H(C^*(H))$  has a norm separable dual Banach space,  $\overline{A_{\gamma_H}(H)}^{w^*}$ , and hence

$$\text{supp } \gamma_H = \gamma_H(C^*(H))^\wedge$$

is countable.  $\square$

**Corollary 4.4.** *Let  $Z(G)$  denote the centre of  $G$ . If  $G/Z(G)$  is compact, then  $A_{\gamma_G}(G)$  is  $w^*$ -closed in  $B(G)$ .*

*Proof.* For  $z \in Z(G)$ ,  $\gamma_G(z)$  is the identity on  $L^2(G)$ . Thus  $\pi(xZ(G)) = \gamma_G(x)$ ,  $x \in G$ , defines a representation of  $G/Z(G)$ , and therefore  $A_\pi(G/Z(G))$  is  $w^*$ -closed in  $B(G/Z(G))$  by Lemma 4.1. Denoting by  $q : G \rightarrow G/Z(G)$  the quotient homomorphism, we have

$$A_{\gamma_G}(G) = A_\pi(G/Z(G)) \circ q.$$

By [1, (2.10)],  $A_{\gamma_G}(G)$  is  $w^*$ -closed in  $B(G)$ .  $\square$

Our goal is to establish the converse to Corollary 4.4 for Lie groups with countably many connected components (Theorem 4.8). Apart from using various results from [21], a major step in proving the theorem will be the next lemma.

We remind the reader that a group  $G$  is called an FC-group if all its conjugacy classes are finite. Such a group, more generally every locally compact group all of whose conjugacy classes are relatively compact, is amenable.

**Lemma 4.5.** *Let  $G$  be a countable discrete FC-group. If  $\text{supp } \gamma_G$  is countable, then  $G$  has a finite commutator subgroup.*

*Proof.* Let  $S = \text{supp } \gamma_G$  and notice first that points in  $S$  are closed in  $\widehat{G}$ . Indeed, the primitive ideal space of any FC-group is a  $T_1$  space (see [28, Theorem 5.2]) and  $C_\gamma^*(G)$ , being a separable  $C^*$ -algebra with countable dual, is of type I. Thus the points of  $S$  are closed in  $\widehat{G}$ . Since  $C^*(G)$  is unital, it follows that every  $\sigma \in S$  is finite dimensional.

Next, employing the facts that points in  $S$  are closed, that  $S$  is countable and that duals of  $C^*$ -algebras are Baire spaces [5, (3.4.13)], a straightforward argument yields the existence of some dense subset  $D$  of  $S$  consisting of points that are also open in  $S$ .

Since  $G$  is a countable amenable group,

$$\bigcup_{\pi \in \widehat{G}} \text{supp}(\pi \otimes \bar{\pi})$$

is a dense subset of  $S$  by [19, Theorem]. Let  $\mathcal{C}(S)$  denote the set of all closed subsets of  $S$ , endowed with Fell's topology [10, p. 427]. By [18, Proposition 2], the mapping

$$\pi \rightarrow \text{supp}(\pi \otimes \bar{\pi}), \widehat{G} \rightarrow \mathcal{C}(S)$$

is continuous. It follows that

$$V = \{\pi \in \widehat{G} : \text{supp}(\pi \otimes \bar{\pi}) \cap D \neq \emptyset\}$$

is non-empty and open in  $\widehat{G}$ .

Now, as points in  $D$  are open in  $S$ ,  $\text{supp}(\pi \otimes \bar{\pi})$  contains a finite dimensional subrepresentation for each  $\pi \in V$ . However,  $\pi \otimes \bar{\pi}$  then also contains the trivial representation  $1_G$ . This can be seen as follows. Suppose that  $\tau$  is finite dimensional and that  $\tau \leq \pi \otimes \bar{\pi}$ . Then

$$1_G \leq \tau \otimes \bar{\tau} \leq \pi \otimes \overline{\pi \otimes \tau},$$

and  $\pi \otimes \tau$  is a (finite) direct sum of irreducible representations  $\rho_1, \dots, \rho_n$ . Thus  $1_G \leq \pi \otimes \bar{\rho}_i$  for some  $i$ , which is impossible unless  $\rho_i \sim \pi$  (see [17, Proposition 2.4]). As is well-known,  $1_G \leq \pi \otimes \bar{\pi}$  forces  $\pi$  to be finite dimensional. Hence every  $\pi \in V$  is finite dimensional.

Finally, for a discrete FC-group  $G$ , the existence of a non-empty open subset in  $\widehat{G}$  consisting of finite dimensional representations implies that  $G$  has a finite commutator subgroup. In fact, in this case the left regular representation of  $G$  has a subrepresentation of type I, and then the commutator subgroup has to be finite by [16, Satz 1] (see also [32, Theorem 3]).  $\square$

**Lemma 4.6.** *Let  $G$  be a locally compact group with open centre. Then, for each  $a \in G$ ,*

$$\text{ind}_{C(a)}^G 1_{C(a)} \prec \gamma_G,$$

where  $C(a)$  denotes the centralizer of  $a$  in  $G$ .

*Proof.* Since  $\text{ind}_{C(a)}^G 1_{C(a)}$  is the cyclic representation of  $G$  defined by the positive definite function  $\chi_{C(a)}$ , the characteristic function of  $C(a)$ , and since  $C(a)$  is open, it suffices to show that given any compact subset  $K$  of  $G \setminus C(a)$ , there is a positive definite function  $\varphi$  associated to  $\gamma_G$  such that  $\varphi(x) = 1$  for all  $x \in C(a)$  and  $\varphi(x) = 0$  for all  $x \in K$ . Now, since

$$C = \{a^{-1}x^{-1}ax : x \in K\}$$

is compact and  $e \notin C$ , we find an open neighborhood  $V$  of  $e$ , contained in the centre of  $G$ , such that  $CV \cap V = \emptyset$ . Let

$$\varphi(x) = |V|^{-1} \langle \gamma_G(x) \chi_{aV}, \chi_{aV} \rangle$$

for  $x \in G$ . Then it is easily verified that  $\varphi(x) = 1$  for all  $x \in C(a)$  and  $\varphi(x) = 0$  for all  $x \in K$ .  $\square$

**Lemma 4.7.** *Suppose that  $G$  is second countable and contains an open normal subgroup  $N$  such that  $G/N$  is abelian and every irreducible representation of  $N$  is finite dimensional. If  $\text{supp } \gamma_G$  is countable, then every irreducible representation of  $G$  is finite dimensional.*

*Proof.* Given an irreducible representation  $\pi$  of  $G$ , there exist a subgroup  $H$  of  $G$  and a finite dimensional irreducible representation  $\tau$  of  $H$  such that  $N \subseteq H$  and

$$\pi \sim \text{ind}_H^G \tau.$$

In fact, this has been shown in [7, Theorem 3.2.3] as an application of representation theory of crossed product  $C^*$ -algebras. We prefer to outline a direct argument for this within the framework of Mackey's unitary representation theory of group extensions [26].

Choose  $\gamma \in \widehat{N}$  such that  $\pi|_N \sim G(\gamma)$ , the  $G$ -orbit of  $\gamma$  in  $\widehat{N}$  under the action of  $G$ . Let  $S$  denote the stability subgroup of  $\gamma$ . By [26, Theorems 8.2 and 8.3], there

exist a multiplier  $\omega$  on  $S/N$ , an irreducible  $\omega$ -representation  $\rho$  of  $S$  in  $\mathcal{H}_\gamma$  and an irreducible  $\omega$ -representation  $\sigma$  of  $S/N$  such that

$$\pi = \text{ind}_S^G(\rho \otimes \sigma).$$

Now, an irreducible  $\omega$ -representation of an abelian group is weakly equivalent to the  $\omega$ -representation induced from some one-dimensional  $\omega$ -representation of a certain subgroup. Hence there exist a subgroup  $H$  of  $S$  containing  $N$  and a one-dimensional  $\omega$ -representation  $\lambda$  of  $H/N$  such that

$$\rho \otimes \sigma \sim \rho \otimes \text{ind}_H^S \lambda = \text{ind}_H^S(\rho|_H \otimes \lambda).$$

Let  $\pi = \rho|_H \otimes \lambda$ , a finite dimensional ordinary representation. Then

$$\pi \sim \text{ind}_S^G(\text{ind}_H^S(\rho|_H \otimes \lambda)) = \text{ind}_H^G \tau,$$

as required.

It follows that

$$\begin{aligned} \pi \otimes \bar{\pi} &= \text{ind}_H^G(\tau \otimes \bar{\tau}|_H) \succ \text{ind}_H^G(\tau \otimes \bar{\tau}) \\ &\succ \text{ind}_H^G 1_H \sim \widehat{G/H}. \end{aligned}$$

On the other hand,  $\pi \otimes \bar{\pi}$  is weakly contained in  $\gamma_G$  since  $G$  is amenable. Now, as  $\text{supp } \gamma_G$  is countable and  $H$  is open and  $G/H$  is abelian,  $H$  must have finite index in  $G$ . Thus,  $\text{ind}_H^G \tau$  is finite dimensional and hence so is  $\pi$ .  $\square$

**Theorem 4.8.** *Suppose that  $G$  is a Lie group with countably many connected components. If  $A_{\gamma_G}(G)$  is  $w^*$ -closed in  $B(G)$ , then  $G/Z(G)$  is compact.*

*Proof.* For any normal subgroup  $N$  of  $G$ , let  $q_N : G \rightarrow G/N$  denote the quotient homomorphism. Since  $A_{\gamma_G}(G)$  is  $w^*$ -closed in  $B(G)$  and  $G$  is second countable, and since the connected component  $G_0$  of  $e$  is open,  $\text{supp } \gamma_{G_0}$  is countable by Corollary 4.3. Theorem 3.4 of [21] yields that  $G_0 = V \times C$ , the direct product of a vector group  $V$  and a compact group  $C$ . Clearly,  $C$  is normal in  $G$ .

We claim that  $V = G_0/C$  is contained in the centre of  $G/C$ . To that end, fix  $a \in G$  and consider the open subgroup  $H$  of  $G$  generated by  $a$  and  $G_0$ . Then  $\text{supp } \gamma_H$  is countable (Corollary 4.3), and since

$$\gamma_{G/C} \circ q_C \prec \gamma_H$$

due to the compactness of  $C$  [19, Remark 1], it follows that  $\gamma_{H/C}$  has a countable support. Since  $G_0/C$  is a vector group and  $(H/C)/(G_0/C) = H/G_0$  is abelian, an application of Lemma 3.2 in [21] shows that  $G_0/C$  is in the centre of  $H/C$ . Since  $a$  is arbitrary,  $G_0/C$  is central in  $G/C$ .

Next we are going to prove that  $G/G_0$  has a finite commutator subgroup. Passing to  $F = G/C$ , we know that  $\gamma_F$  has countable support, and since  $V$  is central in  $F$ , by [21, Lemma 1.1]

$$\gamma_{F/V} \circ q_V \prec \gamma_F,$$

so that  $\text{supp } \gamma_{F/V}$  is countable. Let  $D = F/V$  and denote by  $D_f$  the finite conjugacy class subgroup of  $D$ . Then, by [20, Theorem 1.8]

$$\lambda_{D/D_f} \circ q_{D_f} \prec \gamma_D,$$

where  $\lambda_{D/D_f}$  is the regular representation of  $D/D_f$ . Thus the reduced dual of  $D/D_f$  is countable, and hence  $D/D_f$  is finite (see [2, 34]). At this stage we know in particular that  $G$  is amenable, so that  $\gamma_{G/N} \circ q_N \prec \gamma_G$  for each closed normal

subgroup  $N$  of  $G$  [21, Lemma 1.1]. Also,  $\gamma_{D_f}$  has countable support and therefore the commutator subgroup  $D'_f$  of  $D_f$  is finite by Lemma 4.5. Let  $N$  be the inverse image of  $D'_f$  in  $G$  and let  $E = G/N$ . Then  $\gamma_E$  has countable support, and  $E$  possesses an abelian normal subgroup  $A$  of finite index, namely  $A = D_f/D'_f$ . We apply Lemma 4.6 to  $E$  and obtain that

$$\text{ind}_{C(x)}^E 1_{C(x)} \prec \gamma_E$$

for every  $x \in E$ . It follows that

$$\text{ind}_{C(x) \cap A}^A 1_{C(x) \cap A} \prec (\text{ind}_{C(x)}^E 1_{C(x)})|_A \prec \gamma_E|_A,$$

and  $\gamma_E|_A$  has countable support since  $\gamma_E$  does and  $E/A$  is finite. However,  $A$  being abelian, countability of

$$(A/C(x) \cap A)^\wedge = \text{supp}(\text{ind}_{C(x) \cap A}^A 1_{C(x) \cap A})$$

implies that  $C(x) \cap A$  is of finite index in  $A$ . Hence  $C(x)$  has finite index in  $E$  for every  $x \in E$ . Therefore  $G/N$  is an FC-group, and Lemma 4.5 implies that  $E$  has a finite commutator subgroup. Recalling that  $N/G_0 = D'_f$  is finite, we conclude that  $G/G_0$  has a finite commutator subgroup.

Since  $G/G_0$  has a finite commutator subgroup, there exists a normal subgroup  $N$  of  $G$  such that  $G_0 \subseteq N$ ,  $N/G_0$  is finite and  $G/N$  is abelian. Now,  $G_0 = V \times C$ , and this implies that all the irreducible representations of  $N$  are finite dimensional. Since  $\text{supp } \gamma_G$  is countable, Lemma 4.7 shows that  $G$  has only finite dimensional irreducible representations.

On the other hand,  $G$  has a relatively compact commutator subgroup. To see this, notice first that since  $V$  is contained in the centre of  $N/C$ ,  $N/C$  has a finite commutator subgroup. Denoting its inverse in  $G$  by  $K$ ,  $N/K$  is isomorphic to  $V$ . By arguments that have previously been used,  $V$  is central in  $G/K$ , and applying Lemma 4.6 again, this time to  $G/K$ , we conclude that  $G/K$  is a group with finite conjugacy classes.

Thus  $G$  is a group with relatively compact conjugacy classes all of whose irreducible representations are finite dimensional. Since, in addition,  $G$  is a Lie group, combining Theorem 2 of [27] and Lemma 5.4 of [24] shows that  $G/Z(G)$  is compact.  $\square$

## 5. $w^*$ -CONTINUITY OF THE RESTRICTION MAP

Let  $G$  be a locally compact group, and let  $H$  be a closed subgroup of  $G$ . In this section we study the question of when the restriction map

$$\Phi : B(G) \rightarrow B(H), \varphi \rightarrow \varphi|_H$$

is continuous for the  $w^*$ -topologies. Clearly, if  $H$  is open then  $C^*(H)$  is a subalgebra of  $C^*(G)$  and hence  $\Phi$  is  $w^*$ -continuous. We are going to establish the following converse.

**Theorem 5.1.** *Let  $H$  be a closed subgroup of the locally compact group  $G$ . If the restriction map  $B(G) \rightarrow B(H)$  is continuous for the  $w^*$ -topologies, then  $H$  is open in  $G$ .*

Let  $M(G)$  denote the algebra of all bounded measures on  $G$ . The canonical embedding of  $L^1(G)$  into  $M(G)$  extends to an isometric  $*$ -homomorphism of  $C^*(G)$

into  $C^*(M(G))$ , the enveloping  $C^*$ -algebra of  $M(G)$ . Therefore we may (and shall) identify  $C^*(G)$  with a closed two-sided ideal of  $C^*(M(G))$ .

Let  $H$  be a closed subgroup of  $G$ . For  $f \in L^1(H)$ , let  $\mu_f \in M(G)$  denote the measure on  $G$  defined by  $f$ . The mapping  $f \rightarrow \mu_f$  from  $L^1(H)$  into  $M(G)$  extends to a  $*$ -homomorphism  $\Psi : C^*(H) \rightarrow C^*(M(G))$ . By general principles, the restriction map  $\Phi : B(G) \rightarrow B(H)$  is  $w^*$ -continuous if and only if  $\Phi$  is the transpose of some continuous linear mapping  $\Theta : C^*(H) \rightarrow C^*(G)$ . It is easy to verify that, in this case,  $\Theta$  has to agree with  $\Psi$  on  $L^1(H)$ . Hence we have the following lemma.

**Lemma 5.2.** *The restriction map  $\Phi : B(G) \rightarrow B(H)$  is  $w^*$ -continuous if and only if the range of  $\Psi : C^*(H) \rightarrow C^*(M(G))$  is contained in  $C^*(G)$ .*

*Remark 5.3.* It is interesting to notice that the homomorphism  $\Psi : C^*(H) \rightarrow C^*(M(G))$  need not always be injective (see [4]).

Now, let  $\rho$  denote the right regular representation of  $G$  on  $L^2(G)$  and  $C_r^*(M(G))$  the  $C^*$ -subalgebra of  $\mathcal{L}(L^2(G))$  generated by the set of all operators  $\rho(\mu)$ ,  $\mu \in M(G)$ . The mapping  $f \rightarrow \mu_f$  from  $L^1(H)$  into  $M(G)$  extends to a  $*$ -homomorphism

$$\Psi_r : C^*(H) \rightarrow C_r^*(M(G)).$$

It is clear that  $\Psi_r$  is the composition of  $\Psi$  and the canonical homomorphism from  $C^*(M(G))$  onto  $C_r^*(M(G))$ .

In view of Lemma 5.2, we thus observe that Theorem 5.1 will be a consequence of the following stronger result.

**Theorem 5.4.** *Let  $H$  be a closed subgroup of the locally compact group  $G$ . If the range of the homomorphism  $\Psi_r : C^*(H) \rightarrow C_r^*(M(G))$  is contained in  $C_r^*(G)$ , then  $H$  is open in  $G$ .*

The proof of Theorem 5.4 depends on two elementary lemmas, the first of which appears also in [25]. For the sake of completeness, however, we give a very short and different proof.

**Lemma 5.5.** *Let  $G$  be a locally compact group and let  $T \in C_r^*(G)$ . Suppose that  $K$  is a compact subset of  $G$  and regard  $L^2(K)$  as a closed subspace of  $L^2(G)$  in the usual manner. Then the restriction*

$$T|_{L^2(K)} : L^2(K) \rightarrow L^2(G)$$

*of  $T$  to  $L^2(K)$  is a compact operator.*

*Proof.* Of course, it suffices to prove the statement for operators  $T$  of the form  $T = \rho(f)$  where  $f \in C_c(G)$ . Let  $K' = \text{supp } f \cdot K$ . Then  $Tg \in L^2(K')$  for all  $g \in L^2(K)$ . Choose  $f_j \in C_c(K')$  and  $g_j \in C_c(K)$  such that, as  $n \rightarrow \infty$ ,

$$\int_G \int_G \left| \sum_{j=1}^n f_j(x)g_j(y) - f(xy) \right|^2 dx dy \rightarrow 0.$$

It is straightforward to verify that this implies that  $T : L^2(K) \rightarrow L^2(K')$  is a norm limit of finite rank operators.  $\square$

**Lemma 5.6.** *Let  $(X, \mu)$  be a probability measure space without atoms. Then the canonical embedding  $L^2(X) \rightarrow L^1(X)$  fails to be compact.*

*Proof.* It suffices to show that there is an orthonormal sequence  $(f_n)_n$  in  $L^2(X)$  such that  $\|f_n\|_\infty \leq 1$ . Indeed, we then have

$$2 = \|f_n - f_m\|_2^2 = \int_X |f_n(x) - f_m(x)|^2 d\mu(x) \leq \|f_n - f_m\|_1$$

for all  $n, m \in \mathbb{N}$ ,  $n \neq m$ , and hence no subsequence of  $(f_n)_n$  can converge in  $L^1(X)$ .

Since  $\mu$  has no atoms, there exists, for every  $0 \leq r \leq 1$ , a measurable subset  $A$  of  $X$  with  $\mu(A) = r$  (see [13, Section 4.1, Exercise 1]). Therefore, we can choose inductively, for every  $n \in \mathbb{N}$ , disjoint measurable subsets

$$A_1^{(n)}, A_2^{(n)}, \dots, A_{2^{n-1}}^{(n)}$$

of  $X$  with the following properties:

- (1)  $\mu(A_i^{(n)}) = \frac{1}{2^{n-1}}$  for  $i = 1, \dots, 2^{n-1}$ .
- (2)  $A_{2i-1}^{(n+1)}, A_{2i}^{(n+1)} \subseteq A_i^{(n)}$  for  $i = 1, \dots, 2^{n-1}$ .

For each  $n \in \mathbb{N}$ , define a function  $f_n \in L^2(X)$  (a kind of Rademacher function) by setting

$$f_n(x) = (-1)^i \text{ for } x \in A_i^{(n)} \text{ and } i = 1, \dots, 2^{n-1}.$$

By construction and (1) and (2), we obviously have

$$\|f_n\|_\infty = \|f_n\|_2 = 1 \text{ and } \int_X f_n(x)f_m(x)d\mu(x) = 0, \quad n \neq m.$$

This completes the proof.  $\square$

*Proof of Theorem 5.4.* Let  $H$  be a closed subgroup of  $G$  and suppose  $\Psi_r(C^*(H))$  is contained in  $C_r^*(G)$ . Fix a Bruhat function  $\beta$  on  $G$  for  $H$  (see [31, Chapter 8]), that is, a non-negative continuous function  $\beta$  on  $G$  with the following properties:

- (i) For every compact subset  $K$  of  $G$ ,  $\beta$  agrees on  $KH$  with the restriction of some function from  $C_c(G)$ .
- (ii)  $\int_H \beta(xh)dh = 1$  for all  $x \in G$ .

Let  $\Delta_G$  and  $\Delta_H$  denote the modular functions of  $G$  and  $H$ , respectively, and define a function  $q$  on  $G$  by

$$q(x) = \int_H \beta(xh)\Delta_G(h)\Delta_H(h)^{-1}dh.$$

Since  $q(xh) = q(x)\Delta_G(h)^{-1}\Delta_H(h)$  for all  $x \in G$  and  $h \in H$ , there exists a quasi-invariant measure  $d_q\dot{x}$  on  $\dot{G} = G/H$  such that

$$\int_G \left( \frac{f(xh)}{q(xh)} dh \right) d_q\dot{x} = \int_G f(x)dx$$

for all  $f \in L^1(G)$  [31, Chapter 8].

Now let  $\dot{K} \subseteq \dot{G}$  be a compact neighbourhood of  $\dot{e}$  in  $\dot{G}$ . We are going to prove that  $\dot{K}$  is finite. Clearly, this will imply that  $H$  is open. Let  $\pi : G \rightarrow \dot{G}$  denote the quotient map and choose a compact neighbourhood  $K$  of  $e$  in  $G$  so that  $\pi(K) = \dot{K}$ . Fix a continuous function  $f$  with compact support on  $H$  such that

$$\int_H f(h)\Delta_G(h)^{-1}dh \neq 0.$$

Let  $K' = KH \cap \text{supp } \beta$ , which is a compact subset of  $G$ . By Lemma 5.5, the restriction of  $\rho(\mu_f)$ , the right convolution operator defined by  $\mu_f$ , to  $L^2(K')$  is a compact operator. Define a linear mapping

$$T : L^2(\dot{K}, d_q \dot{x}) \rightarrow L^2(K')$$

by  $T\dot{g}(x) = \dot{g}(\dot{x})\beta(x)^{1/2}$  for  $\dot{g} \in L^2(\dot{K}, d_q \dot{x})$  and  $x \in K$ . Since

$$\int_H \frac{\beta(xh)}{q(xh)} dh = \frac{1}{q(x)} \int_H \beta(xh) \Delta_G(h) \Delta_H(h)^{-1} dh = 1$$

for all  $x \in G$ , we have

$$\begin{aligned} \|T\dot{g}\|_2^2 &= \int_{\dot{G}} \left( \int_H \frac{|\dot{g}(\dot{x})|^2 \beta(xh)}{q(xh)} dh \right) d_q \dot{x} \\ &= \int_{\dot{G}} |\dot{g}(\dot{x})|^2 d_q \dot{x} = \|\dot{g}\|_2^2. \end{aligned}$$

Thus  $T$  is a bounded linear operator. Set

$$q_1(x) = \int_H \beta(xh)^{1/2} \Delta_G(h) \Delta_H(h)^{-1} dh$$

for  $x \in G$ . Then, for all  $x \in G$  and  $h \in H$ ,

$$q_1(xh) = q_1(x) \Delta_G(h)^{-1} \Delta_H(h).$$

Hence there exists a quasi-invariant measure  $d_{q_1} \dot{x}$  on  $\dot{G}$  such that

$$\int_{\dot{G}} \left( \int_H \frac{q(xh)}{q_1(xh)} dh \right) d_{q_1} \dot{x} = \int_G g(x) dx$$

for all  $g \in L^1(G)$ . Let

$$S : L^1(G) \rightarrow L^1(\dot{G}, d_{q_1} \dot{x})$$

be the linear operator defined by

$$Sg(\dot{x}) = \int_H \frac{g(xh)}{q_1(xh)} dh$$

for  $g \in C_c(G)$ .  $S$  is bounded since

$$\|Sg\|_{L^1(\dot{G}, d_{q_1} \dot{x})} \leq \int_{\dot{G}} \left( \int_H \frac{|g(xh)|}{q_1(xh)} dh \right) d_{q_1} \dot{x} = \|g\|_1.$$

Observe next that  $\rho(\mu_f)$  maps  $L^2(K')$  into  $L^2(K'')$  for some compact subset  $K''$  of  $G$ . Hence  $\rho(\mu_f)T$  map  $L^2(\dot{K}, d_q \dot{x})$  into  $L^1(G)$ , and we may consider the bounded linear operator

$$S\rho(\mu_f)T : L^2(\dot{K}, d_q \dot{x}) \rightarrow L^1(\dot{G}, d_{q_1} \dot{x}).$$

For  $\dot{g} \in C(\dot{K})$ , we have

$$\begin{aligned} S\rho(\mu_{f|_H})T(\dot{g})(\dot{x}) &= \dot{g}(\dot{x}) \int_H \frac{1}{q_1(xh)} \left( \int_H \beta(xhk)^{1/2} f(k) dk \right) dh \\ &= \dot{g}(\dot{x}) \int_H f(k) \left( \int_H \frac{\beta(xhk)^{1/2}}{q_1(xh)} dh \right) dk. \end{aligned}$$

However, for  $x \in G$  and  $k \in K$ ,

$$\begin{aligned} \int_H \frac{\beta(xhk)^{1/2}}{q_1(xh)} dh &= \Delta_H(k)^{-1} \int_H \frac{\beta(xh)^{1/2}}{q_1(xhk^{-1})} dh \\ &= \Delta_G(k)^{-1} \int_H \frac{\beta(xh)^{1/2}}{q(xh)} dh = \Delta_G(k)^{-1}. \end{aligned}$$

Thus, setting  $\alpha = \int_H f(k) \Delta_G(k)^{-1} dk \neq 0$ , we obtain that

$$S\rho(\mu_{f|H})T(\dot{g}) = \alpha\dot{g}$$

for all  $\dot{g} \in L^2(\dot{K}, d_q\dot{x})$ . Now, by hypothesis and Lemma 5.5, the restriction of  $\rho(\mu_{f|H})$  to  $L^2(K')$  is compact. Hence

$$\alpha I : L^2(\dot{K}, d_q\dot{x}) \rightarrow L^1(\dot{K}, d_{q_1}\dot{x})$$

is a compact operator.

Finally, notice that since

$$d_{q_1}\dot{x} = \frac{q_1(x)}{q(x)} d_q\dot{x}$$

and  $\frac{q_1}{q}$  is a strictly positive continuous function on the compact set  $\dot{K}$ , the corresponding  $L^1$ -spaces are equal and the  $L^1$ -norms are equivalent. Hence the embedding

$$L^2(\dot{K}, d_q\dot{x}) \rightarrow L^1(\dot{K}, d_q\dot{x})$$

is compact. By Lemma 5.6,  $\dot{K}$  has to have atoms. However, this implies that  $\dot{K}$  is finite.  $\square$

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